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ScienceDirect
J. Differential Equations 230 (2006) 446–464
Journal of Differential Equations
www.elsevier.com/locate/jde

Krein's formula for indefinite multipliers in linear periodic Hamiltonian systems

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Received 30 March 2005

Abstract

This paper is concerned with Hamiltonian systems of linear differential equations with periodic coefficients under a small perturbation. It is well known that Krein's formula determines the behavior of definite multipliers on the unit circle and is quite useful in studying the (strong) stability of Hamiltonian system. Our aim is to give a simple formula that determines the behavior of indefinite multipliers with two multiplicity, which is generic case. The result does not require analyticity and is proved directly. Applying this formula, we obtain instability criteria for solutions with periodic structure in nonlinear dissipative systems such as the Swift–Hohenberg equation and reaction–diffusion systems of activator–inhibitor type.

Keywords: Krein's formula; Indefinite multiplier; Hamiltonian system

1. Introduction

Let us consider Hamiltonian systems of linear differential equations with real periodic coefficients under a small perturbation:

$$Jx_t = H(t,\varepsilon)x, \quad x \in \mathbf{R}^{2n}, \tag{1.1}$$

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2006.08.005

where J is a skew-symmetric matrix of the form

$$J = \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix} \tag{1.2}$$

with a nonsingular symmetric matrix S, and $H(t,\varepsilon)$ is continuous with respect to t and a real symmetric matrix with

$$H(t+T,\varepsilon) = H(t,\varepsilon) \tag{1.3}$$

for some period $T > 0$. Let $M(t,\varepsilon)$ denote the fundamental solution matrix for (1.1), that is, $M(t,\varepsilon)$ is a matrix-valued function satisfying

$$\begin{cases} JM_t(t,\varepsilon) = H(t,\varepsilon)M(t,\varepsilon), \\ M(0,\varepsilon) = I_{2n}. \end{cases} \tag{1.4}$$

In particular, $M(T,\varepsilon)$ is called the monodromy matrix of (1.1). The eigenvalues of $M(T,\varepsilon)$, denoted by $\mu_j(\varepsilon)$ $(j = 1,2,\ldots,2n)$, are called *multipliers*, and a multiplier $\mu_j(\varepsilon)$ is called *definite* if it is a simple eigenvalue of the monodromy matrix, i.e., the algebraic multiplicity is equal to the geometric multiplicity. It is well known that behavior of multipliers on the unit circle in $\mathbf{C}$ essentially determines (strong) stability of Hamiltonian systems (see Section 3.1 and Arnold [2], Ekeland [4], Yakubovich and Starzhinskii [16], etc.). As for the definite multipliers on the unit circle, a simple formula derived by Krein gives useful information about their behavior.

Theorem 1.1. [16, Theorem, p. 167] *Let $H(t,\varepsilon)$ be a C^1-function with respect to ε. Assume that ρ $(|\rho| = 1)$ is an r-fold $(r \geqslant 1)$ definite multiplier of $M(T,0)$, and let η_j be an associated eigenvector of $\mu_j(0) = \rho$ and $\eta_j(t) := M(t,0)\eta_j$. Then $\langle J\eta_j,\eta_j\rangle \neq 0$ and*

$$\left.\frac{d\mu_j(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \frac{\rho}{\langle J\eta_j,\eta_j\rangle}\int_0^T \left\langle \frac{\partial H}{\partial\varepsilon}(t,0)\eta_j(t),\eta_j(t)\right\rangle dt, \tag{1.5}$$

where $\langle\ ,\ \rangle$ denotes the usual inner product in $\mathbf{C}^n$.

From this result, when ρ $(= \mu_j(0))$ is an r-fold definite multiplier, the behavior of $\mu_j(\varepsilon)$ for sufficiently small ε is determined by (1.5) together with the property of real symplectic matrix that μ^{-1} and $\bar{\mu}$ are eigenvalues whenever μ is. In other words, (1.5) provides a theoretical basis for the study of (strong) stability of Hamiltonian systems known as Krein's theory [4,8,16], that was further extended by Gelfand and Lidsky [6], and independently by Moser [14].

On the other hand, when ρ is an *indefinite* multiplier (i.e., the algebraic multiplicity is greater than the geometric multiplicity), a formula such as (1.5) has not been known. As seen in [16], the behavior of indefinite multipliers on the unit circle was studied by rather involved arguments based on the theory of analytic functions under the assumption that $H(t,\varepsilon)$ is analytic in ε and, as a consequence, we have not yet arrived at clear understanding for their behavior.

In this paper, we are interested in the behavior of indefinite multipliers with two multiplicity (i.e., the algebraic multiplicity is equal to 2 while the geometric multiplicity is 1). It is the most typical and generic case, especially for Hamiltonian systems with two degree of freedom. In what follows, we suppose that $\mu_1(\varepsilon)$ and $\mu_2(\varepsilon)$ are simple for $\varepsilon \neq 0$, and that

$$\mu_1(\varepsilon) \neq \mu_2(\varepsilon) \quad \text{for } \varepsilon \neq 0, \tag{1.6}$$

because we often encounter with a situation in which two multipliers collide at some point on the unit circle when $\varepsilon = 0$ in many practical problems as seen in Section 3. Our main result, stated in Theorem 1.2, gives a simple formula for the behavior of multipliers when ρ is an indefinite multiplier with two multiplicity under the assumption (1.6). We note that $\mu_1(\varepsilon)+\mu_2(\varepsilon)$ is differentiable in ε (see Kato [7]), though neither $\mu_1(\varepsilon)$ nor $\mu_2(\varepsilon)$ is differentiable.

Theorem 1.2. *Let $H(t,\varepsilon)$ be a C^1-function with respect to ε. Assume that* (1.6) *and ρ $(= \mu_1(0) = \mu_2(0))$ is an indefinite multiplier. Then*

$$\frac{d}{d\varepsilon}\big(\mu_1(\varepsilon)+\mu_2(\varepsilon)\big)\Big|_{\varepsilon=0} = -\frac{\bar{\rho}}{\langle J\eta_1, \bar{\eta}_2\rangle}\int_0^T \left\langle \frac{\partial H}{\partial \varepsilon}(t,0)\eta_1(t),\, \overline{\eta_1(t)}\right\rangle dt, \tag{1.7}$$

where η_1 is an eigenvector associated with ρ, η_2 is a generalized eigenvector satisfying $M(T,0)\eta_2 = \rho\eta_2 + \rho\eta_1$, and $\eta_1(t) := M(t,0)\eta_1$.

Remark 1. Formula (1.7) can be rewritten as follows:

$$\frac{d}{d\varepsilon}\big(\mu_1(\varepsilon)+\mu_2(\varepsilon)\big)\Big|_{\varepsilon=0} = -\bar{\rho}T\frac{\int_0^T \langle \frac{\partial H}{\partial \varepsilon}(t,0)\eta_1(t), \overline{\eta_1(t)}\rangle\, dt}{\int_0^T \langle J\eta_1(t), \overline{\eta_2(t)}\rangle\, dt}. \tag{1.8}$$

Indeed, it is easy to see that $\langle J\eta_1(t), \overline{\eta_2(t)}\rangle = \langle J\eta_1, \bar{\eta}_2\rangle$ for all t and hence

$$\langle J\eta_1, \bar{\eta}_2\rangle = \frac{1}{T}\int_0^T \big\langle J\eta_1(t), \overline{\eta_2(t)}\big\rangle dt,$$

which yields (1.8). Similarly, (1.5) can be also rewritten as

$$\frac{d\mu_j(\varepsilon)}{d\varepsilon}\bigg|_{\varepsilon=0} = \rho T\frac{\int_0^T \langle \frac{\partial H}{\partial \varepsilon}(t,0)\eta_j(t), \eta_j(t)\rangle\, dt}{\int_0^T \langle J\eta_j(t), \eta_j(t)\rangle\, dt}.$$

Remark 2. Formula (1.5) is formulated in the framework of Hamiltonian systems with complex coefficients

$$i^{-1}Gx_t = H(t,\lambda)x, \quad x \in \mathbf{R}^{2n}, \tag{1.9}$$

where G is a nonsingular Hermitian matrix, and $H(t,\lambda)$ is a Hermitian matrix for $\lambda \in \mathbf{C}$. In fact, it is shown in [16] that

$$\frac{d\mu_j(\lambda)}{d\lambda}\bigg|_{\lambda=0} = \frac{i\rho}{\langle G\eta_j, \eta_j\rangle}\int_0^T \left\langle \frac{\partial H}{\partial \lambda}(t,0)\eta_j(t), \eta_j(t)\right\rangle dt, \quad j=1,2, \tag{1.10}$$

which corresponds to (1.5). However, (1.7) cannot be extended to the complex case (1.9) when $\rho \neq \pm 1$. For $\rho = \pm 1$, a slight modification of the argument given in the next section yields

$$\frac{d}{d\lambda}\big(\mu_1(\lambda) + \mu_2(\lambda)\big)\Big|_{\lambda=0} = -\bar{\rho}\,\mathrm{Re}\left\{ \frac{i}{\langle G\eta_1, \eta_2\rangle} \int_0^T \left\langle \frac{\partial H}{\partial \lambda}(t,0)\eta_1(t), \eta_1(t)\right\rangle dt \right\}. \tag{1.11}$$

As seen in Section 3, (1.11) with the aid of Remark 1 is useful to study Hamiltonian systems associated with a linearized eigenvalue problem of periodic solutions in nonlinear problems.

Remark 3. Formula (1.5) shows that the behavior of definite multiplier μ_j strongly depends on $\langle J\eta_j, \eta_j\rangle$. In fact, as seen in [1,4,16], it plays a central role for the index theory in the study of periodic solutions of Hamiltonian systems when multipliers are definite. Formula (1.7) suggests that the index theory can be extended to the case of indefinite multipliers. That is, we can define the index of indefinite multiplier with two multiplicity by using $\langle J\eta_1, \bar{\eta}_2\rangle$.

The organization of this paper is as follows. In the next section, we give a proof of Theorem 1.2. The proof is direct and does not require analyticity of $H(t,\varepsilon)$ in ε. In Section 3, we give two well-known examples that demonstrate usefulness of Theorem 1.2. We give an extension of the Krein–Lyubarskii theorem [16] that treats the behavior of multipliers. Moreover, it is shown that the Eckhaus instability criteria in dissipative systems with gradient/skew-gradient structure introduced in [10,17] are reformulated in the framework of our results.

2. Proof of Theorem 1.2

In this section, we give a derivation of the formula (1.7). We mainly consider the case of $\rho \neq \pm 1$. The case $\rho = \pm 1$ can be treated in an easier way.

Let us investigate properties of eigenvalues and their associated generalized eigenvectors of $M = M(T,0)$. We denote generalized eigenvectors of $M = M(T,0)$ by η_j $(j = 1, 2, \ldots, 2n)$, and suppose that η_1 and η_2 are generalized eigenvectors associated with a degenerate eigenvalue ρ $(= \mu_1(0) = \mu_2(0))$, that is,

$$M\,(\eta_1\ \eta_2) = (\eta_1\ \eta_2)\begin{pmatrix} \rho & \rho \\ 0 & \rho \end{pmatrix}. \tag{2.1}$$

Without loss of generality, we assume $\langle \eta_1, \eta_2\rangle = 0$. In fact, we set

$$\eta_2 = \tilde{\eta}_2 - \langle \eta_1, \tilde{\eta}_2\rangle \eta_1 \tag{2.2}$$

for $\tilde{\eta}_2$ satisfying $M\tilde{\eta}_2 = \rho\eta_1 + \rho\tilde{\eta}_2$. Then we can easily verify $\langle \eta_1, \eta_2\rangle = 0$ and $M\eta_2 = \rho\eta_1 + \rho\eta_2$. In addition, since M is real, $\eta_3 = \bar{\eta}_1$ and $\eta_4 = \bar{\eta}_2$ are generalized eigenvectors associated with an eigenvalue $\bar{\rho} = \rho^{-1}$, that is,

$$M(\eta_3\ \eta_4) = (\eta_3\ \eta_4)\begin{pmatrix} \rho^{-1} & \rho^{-1} \\ 0 & \rho^{-1} \end{pmatrix}, \tag{2.3}$$

where $\langle \eta_3, \eta_4 \rangle = 0$. Set $W := (\eta_1\ \eta_2\ \eta_3\ \eta_4\ \eta_5 \cdots \eta_{2n})$. Then we have $MW = WA$, where A is of the form

$$A = \left(\begin{array}{cc|cc|c} \rho & \rho & \multicolumn{2}{c|}{O} & O \\ 0 & \rho & & & \\ \hline \multicolumn{2}{c|}{O} & \rho^{-1} & \rho^{-1} & O \\ & & 0 & \rho^{-1} & \\ \hline \multicolumn{2}{c|}{O} & \multicolumn{2}{c|}{O} & B \end{array}\right)$$

and B is a $(2n-4)\times(2n-4)$ matrix. Setting $W^* := {}^{\mathrm t}W^{-1} = (\eta_1^*\ \eta_2^*\ \eta_3^*\ \eta_4^*\ \eta_5^* \ \cdots\ \eta_{2n}^*)$, we immediately see that $\langle \eta_i, \eta_j^* \rangle = \delta_{ij}$ and $M^*W^* = W^*A^*$, where $M^* := {}^{\mathrm t}M^{-1}$ and

$$A^* := {}^{\mathrm t}A^{-1} = \left(\begin{array}{cc|cc|c} \rho^{-1} & 0 & \multicolumn{2}{c|}{O} & O \\ -\rho^{-1} & \rho^{-1} & & & \\ \hline \multicolumn{2}{c|}{O} & \rho & 0 & O \\ & & -\rho & \rho & \\ \hline \multicolumn{2}{c|}{O} & \multicolumn{2}{c|}{O} & {}^{\mathrm t}B^{-1} \end{array}\right).$$

Here we note that $\eta_3^* = \bar\eta_1^*$ and $\eta_4^* = \bar\eta_2^*$.

In the following lemmas, we assume $\rho \neq \pm 1$.

Lemma 2.1. *Suppose that* $\langle J\eta_1, \eta_4 \rangle = 1$. *Then* $\langle J\eta_3, \eta_2 \rangle = 1$, $\langle J\eta_2, \eta_3 \rangle = \langle J\eta_4, \eta_1 \rangle = -1$ *and* $\langle J\eta_i, \eta_j \rangle = 0$ *for other i and j. Moreover,*

$$\eta_1^* = -J\eta_4, \qquad \eta_2^* = J\eta_3, \qquad \eta_3^* = -J\eta_2, \qquad \eta_4^* = J\eta_1.$$

Proof. Since $\bar\eta_1 = \eta_3$ and $\bar\eta_2 = \eta_4$, we have

$$\langle J\eta_3, \eta_2 \rangle = \langle J\bar\eta_1, \bar\eta_4 \rangle = \overline{\langle J\eta_1, \eta_4 \rangle} = 1.$$

Moreover, we have

$$\langle J\eta_2, \eta_3 \rangle = -\langle \eta_2, J\eta_3 \rangle = -\overline{\langle J\eta_3, \eta_2 \rangle} = -1, \qquad \langle J\eta_4, \eta_1 \rangle = -\langle \eta_4, J\eta_1 \rangle = -\overline{\langle J\eta_1, \eta_4 \rangle} = -1.$$

Since M is symplectic with the property that $M^* = {}^{\mathrm t}M^{-1} = JMJ^{-1}$, it follows from $M^*W^* = W^*A^*$ that $MJ^{-1}W^* = J^{-1}W^*A^*$. Comparing this equality with $MW = WA$, we have

$$J^{-1}\eta_2^* = c_1\eta_3, \qquad J^{-1}\eta_4^* = c_2\eta_1$$

for some c_1 and c_2. On the other hand, it follows from $\langle \eta_2, \eta_2^* \rangle = \langle \eta_4, \eta_4^* \rangle = 1$ that $\bar c_1 \langle \eta_2, J\eta_3 \rangle = \bar c_2 \langle \eta_4, J\eta_1 \rangle = 1$. Hence we see that $c_1 = c_2 = 1$ because $\langle \eta_2, J\eta_3 \rangle = \overline{\langle J\eta_3, \eta_2 \rangle} = 1$ and $\langle \eta_4, J\eta_1 \rangle = \overline{\langle J\eta_1, \eta_4 \rangle} = 1$. Furthermore, we can easily verify $\langle J\eta_i, \eta_j \rangle = 0$ for other i and j by using ${}^{\mathrm t}J = -J$, $\langle \eta_k, \eta_2^* \rangle = 0$ $(k \neq 2)$, $\langle \eta_\ell, \eta_2^* \rangle = 0$ $(\ell \neq 4)$, $J^{-1}\eta_2^* = \eta_3$ and $J^{-1}\eta_4^* = \eta_1$.

Next we show $\eta_1^* = -J\eta_4$. From $MW = WA$ and $MJ^{-1}W^* = J^{-1}W^*A^*$, we have $M(-\eta_4) = \rho^{-1}(-\eta_4) - \rho^{-1}\eta_3$ and $M(J^{-1}\eta_1^*) = \rho^{-1}(J^{-1}\eta_1^*) - \rho^{-1}(J^{-1}\eta_2^*) = \rho^{-1}(J^{-1}\eta_1^*) - \rho^{-1}\eta_3$. Therefore, we have

$$J^{-1}\eta_1^* = -\eta_4 + c\eta_3$$

for some c. Since $\langle \eta_2, \eta_1^* \rangle = 0$ and ${}^{\mathrm{t}}J = -J$, we have $\langle J\eta_2, \eta_4 \rangle - \bar{c}\langle J\eta_2, \eta_3 \rangle = 0$, which yields $c = 0$ by virtue of $\langle J\eta_2, \eta_4 \rangle = 0$ and $\langle J\eta_2, \eta_3 \rangle = -1$. In this case, we can easily verify $\langle \eta_1, \eta_1^* \rangle = 1$ and $\langle \eta_3, \eta_1^* \rangle = \langle \eta_4, \eta_1^* \rangle = 0$. Similarly, we can show $\eta_3^* = -J\eta_2$. $\square$

Lemma 2.2. *There exists a basis $\{\xi_1(\varepsilon), \xi_2(\varepsilon), \xi_3(\varepsilon), \xi_4(\varepsilon), \ldots, \xi_{2n}(\varepsilon)\}$ with the following properties. Let*

$$W(\varepsilon) := \big(\xi_1(\varepsilon)\ \xi_2(\varepsilon)\ \xi_3(\varepsilon)\ \xi_4(\varepsilon) \cdots \xi_{2n}(\varepsilon)\big) \quad \textit{and}$$
$$W^*(\varepsilon) := {}^{\mathrm{t}}W(\varepsilon)^{-1} = \big(\xi_1^*(\varepsilon)\ \xi_2^*(\varepsilon)\ \xi_3^*(\varepsilon)\ \xi_4^*(\varepsilon) \cdots \xi_{2n}^*(\varepsilon)\big).$$

The vectors $\xi_1(\varepsilon)$, $\xi_2(\varepsilon)$, $\xi_3(\varepsilon)$ and $\xi_4(\varepsilon)$ are differentiable in ε on $(-\delta, \delta)$ with some $\delta > 0$, and the following equalities hold:

$$\xi_3(\varepsilon) = \bar{\xi}_1(\varepsilon), \qquad \xi_4(\varepsilon) = \bar{\xi}_2(\varepsilon), \tag{2.4}$$

$$\lim_{\varepsilon \to 0} \xi_j(\varepsilon) = \eta_j, \qquad \lim_{\varepsilon \to 0} \xi_j^*(\varepsilon) = \eta_j^*, \tag{2.5}$$

$$\mu_1(\varepsilon) + \mu_2(\varepsilon) = \big\langle M(\varepsilon)\xi_1(\varepsilon), \xi_1^*(\varepsilon)\big\rangle + \big\langle M(\varepsilon)\xi_2(\varepsilon), \xi_2^*(\varepsilon)\big\rangle, \tag{2.6}$$

$$\xi_1^*(\varepsilon) = -J\xi_4(\varepsilon), \qquad \xi_2^*(\varepsilon) = J\xi_3(\varepsilon), \qquad \xi_3^*(\varepsilon) = -J\xi_2(\varepsilon), \qquad \xi_4^*(\varepsilon) = J\xi_1(\varepsilon), \tag{2.7}$$

where $M(\varepsilon) = M(T, \varepsilon)$.

Remark 4. The following example is helpful to understand behavior of degenerate eigenvectors and the proof of Lemma 2.2. Let us consider

$$A(\varepsilon) = \begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix}.$$

When $\varepsilon = 0$, $A(0)$ has a degenerate eigenvalue 1 and its associated generalized eigenvectors $\eta_1 = {}^{\mathrm{t}}(c_1, 0)$ and $\eta_2 = {}^{\mathrm{t}}(0, c_2)$ for some $c_1 > 0$ and $c_2 > 0$. On the other hand, when $\varepsilon \neq 0$, $A(\varepsilon)$ has eigenvalues and their associated eigenvectors as follows:

$$\mu_1(\varepsilon) = \begin{cases} 1 + \sqrt{\varepsilon} & \text{for } \varepsilon > 0, \\ 1 + i\sqrt{-\varepsilon} & \text{for } \varepsilon < 0, \end{cases} \qquad \mu_2(\varepsilon) = \begin{cases} 1 - \sqrt{\varepsilon} & \text{for } \varepsilon > 0, \\ 1 - i\sqrt{-\varepsilon} & \text{for } \varepsilon < 0, \end{cases}$$

$$\hat{\xi}_1(\varepsilon) = \begin{cases} {}^{\mathrm{t}}(1, \sqrt{\varepsilon}\,) & \text{for } \varepsilon > 0, \\ {}^{\mathrm{t}}(1, i\sqrt{-\varepsilon}\,) & \text{for } \varepsilon < 0, \end{cases} \qquad \hat{\xi}_2(\varepsilon) = \begin{cases} {}^{\mathrm{t}}(1, -\sqrt{\varepsilon}\,) & \text{for } \varepsilon > 0, \\ {}^{\mathrm{t}}(1, -i\sqrt{-\varepsilon}\,) & \text{for } \varepsilon < 0. \end{cases}$$

It is clear that if we define

$$\tilde{\xi}_1(\varepsilon) = \|\eta_1\| \frac{\hat{\xi}_1(\varepsilon) + \hat{\xi}_2(\varepsilon)}{\|\hat{\xi}_1(\varepsilon) + \hat{\xi}_2(\varepsilon)\|} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix},$$

then $\tilde{\xi}_1(\varepsilon)$ is differentiable (constant) with respect to ε and satisfies $\lim_{\varepsilon\to 0}\tilde{\xi}_1(\varepsilon)=\eta_1$. However, since

$$\tilde{\xi}_2(\varepsilon)=\|\eta_2\|\,\frac{\hat{\xi}_1(\varepsilon)-\hat{\xi}_2(\varepsilon)}{\|\hat{\xi}_1(\varepsilon)-\hat{\xi}_2(\varepsilon)\|}=\begin{cases}{}^{\mathrm{t}}(0,c_2) & \text{for } \varepsilon>0,\\ {}^{\mathrm{t}}(0,c_2 i) & \text{for } \varepsilon<0,\end{cases}$$

is not differentiable at $\varepsilon=0$, we must set

$$\tilde{\xi}_2(\varepsilon)=\begin{cases}\|\eta_2\|\dfrac{\hat{\xi}_1(\varepsilon)-\hat{\xi}_2(\varepsilon)}{\|\hat{\xi}_1(\varepsilon)-\hat{\xi}_2(\varepsilon)\|} & \text{for } \varepsilon>0,\\ -i\|\eta_2\|\dfrac{\hat{\xi}_1(\varepsilon)-\hat{\xi}_2(\varepsilon)}{\|\hat{\xi}_1(\varepsilon)-\hat{\xi}_2(\varepsilon)\|} & \text{for } \varepsilon<0,\end{cases}$$

so that $\tilde{\xi}_2(\varepsilon)={}^{\mathrm{t}}(0,c_2)$ is differentiable in ε and $\lim_{\varepsilon\to 0}\tilde{\xi}_2(\varepsilon)=\eta_2$.

Proof of Lemma 2.2. We construct such a basis concretely. Let $\{\hat{\xi}_j(\varepsilon)\}$ be generalized eigenvectors of $M(\varepsilon)$ for $\varepsilon\neq 0$ with the properties

$$\hat{\xi}_3(\varepsilon)=\overline{\hat{\xi}_1(\varepsilon)},\qquad \hat{\xi}_4(\varepsilon)=\overline{\hat{\xi}_2(\varepsilon)},\qquad \bigl\|\hat{\xi}_j(\varepsilon)\bigr\|=\|\eta_1\|\ \bigl(=\|\eta_3\|\bigr),\quad j=1,2,\dots,2n,$$

and

$$\lim_{\varepsilon\to 0}\hat{\xi}_1(\varepsilon)=\lim_{\varepsilon\to 0}\hat{\xi}_2(\varepsilon)=\eta_1,\qquad \lim_{\varepsilon\to 0}\hat{\xi}_3(\varepsilon)=\lim_{\varepsilon\to 0}\hat{\xi}_4(\varepsilon)=\eta_3.$$

We set

$$\tilde{\xi}_1(\varepsilon)=\|\eta_1\|\frac{\hat{\xi}_1(\varepsilon)+\hat{\xi}_2(\varepsilon)}{\|\hat{\xi}_1(\varepsilon)+\hat{\xi}_2(\varepsilon)\|},\qquad \tilde{\xi}_2(\varepsilon)=\|\eta_1\|\frac{\hat{\xi}_1(\varepsilon)-\hat{\xi}_2(\varepsilon)}{\|\hat{\xi}_1(\varepsilon)-\hat{\xi}_2(\varepsilon)\|},$$
$$\tilde{\xi}_3(\varepsilon)=\|\eta_1\|\frac{\hat{\xi}_3(\varepsilon)+\hat{\xi}_4(\varepsilon)}{\|\hat{\xi}_3(\varepsilon)+\hat{\xi}_4(\varepsilon)\|},\qquad \tilde{\xi}_4(\varepsilon)=\|\eta_1\|\frac{\hat{\xi}_3(\varepsilon)-\hat{\xi}_4(\varepsilon)}{\|\hat{\xi}_3(\varepsilon)-\hat{\xi}_4(\varepsilon)\|}.$$

Then we have

$$\bigl\|\tilde{\xi}_j(\varepsilon)\bigr\|=\|\eta_1\|,\quad j=1,2,3,4,\qquad \lim_{\varepsilon\to 0}\tilde{\xi}_1(\varepsilon)=\eta_1,\qquad \lim_{\varepsilon\to 0}\tilde{\xi}_3(\varepsilon)=\eta_3,\quad \text{and}$$
$$\lim_{\varepsilon\to 0}\bigl\langle\tilde{\xi}_1(\varepsilon),\tilde{\xi}_2(\varepsilon)\bigr\rangle=\lim_{\varepsilon\to 0}\bigl\langle\tilde{\xi}_3(\varepsilon),\tilde{\xi}_4(\varepsilon)\bigr\rangle=0,$$

so that

$$\lim_{\varepsilon\to 0}\bar{\sigma}(\varepsilon)\tilde{\xi}_2(\varepsilon)=\|\eta_1\|\frac{\eta_2}{\|\eta_2\|},\qquad \lim_{\varepsilon\to 0}\sigma(\varepsilon)\tilde{\xi}_4(\varepsilon)=\|\eta_1\|\frac{\eta_4}{\|\eta_4\|}.$$

Here $\sigma(\varepsilon)\in\mathbf{C}$ $(\varepsilon\neq 0)$ is a continuous function of ε with $|\sigma(\varepsilon)|=1$, which will be specified later. Therefore, if we define $\xi_j(\varepsilon)$ by

$$\xi_1(\varepsilon)=\tilde{\xi}_1(\varepsilon),\qquad \xi_3(\varepsilon)=\tilde{\xi}_3(\varepsilon),\qquad \xi_j(\varepsilon)=\hat{\xi}_j(\varepsilon),\quad j=4,5,\dots,2n,$$
$$\xi_2(\varepsilon)=\bar{\sigma}(\varepsilon)\|\eta_2\|\frac{\hat{\xi}_1(\varepsilon)-\hat{\xi}_2(\varepsilon)}{\|\hat{\xi}_1(\varepsilon)-\hat{\xi}_2(\varepsilon)\|},\qquad \xi_4(\varepsilon)=\sigma(\varepsilon)\|\eta_4\|\frac{\hat{\xi}_3(\varepsilon)-\hat{\xi}_4(\varepsilon)}{\|\hat{\xi}_3(\varepsilon)-\hat{\xi}_4(\varepsilon)\|},$$

then we have (2.4), (2.5),

$$\hat{\xi}_1(\varepsilon)+\hat{\xi}_2(\varepsilon)=a(\varepsilon)\xi_1(\varepsilon),\qquad \hat{\xi}_1(\varepsilon)-\hat{\xi}_2(\varepsilon)=b(\varepsilon)\xi_2(\varepsilon),\quad \text{and}$$
$$\hat{\xi}_3(\varepsilon)+\hat{\xi}_4(\varepsilon)=\bar{a}(\varepsilon)\xi_3(\varepsilon),\qquad \hat{\xi}_3(\varepsilon)-\hat{\xi}_4(\varepsilon)=\bar{b}(\varepsilon)\xi_4(\varepsilon),$$

where

$$a(\varepsilon)=\big\|\hat{\xi}_1(\varepsilon)+\hat{\xi}_2(\varepsilon)\big\|\big/\|\eta_1\|,\qquad b(\varepsilon)=\big\|\hat{\xi}_1(\varepsilon)-\hat{\xi}_2(\varepsilon)\big\|\big/\big(\bar{\sigma}(\varepsilon)\|\eta_2\|\big).$$

Notice that $\lim_{\varepsilon\to0}a(\varepsilon)=2$ and $\lim_{\varepsilon\to0}b(\varepsilon)=0$. Moreover, we have

$$M(\varepsilon)\xi_1(\varepsilon)=\frac{\mu_1(\varepsilon)+\mu_2(\varepsilon)}{2}\cdot\xi_1(\varepsilon)+\nu(\varepsilon)\cdot\frac{\mu_1(\varepsilon)-\mu_2(\varepsilon)}{2}\cdot\xi_2(\varepsilon),\tag{2.8}$$

where $\nu(\varepsilon)=b(\varepsilon)/a(\varepsilon)$. In fact,

$$\begin{aligned}M(\varepsilon)\xi_1(\varepsilon)&=\frac{1}{a(\varepsilon)}M(\varepsilon)\big(\hat{\xi}_1(\varepsilon)+\hat{\xi}_2(\varepsilon)\big)\\&=\frac{1}{a(\varepsilon)}\big(\mu_1(\varepsilon)\hat{\xi}_1(\varepsilon)+\mu_2(\varepsilon)\hat{\xi}_2(\varepsilon)\big)\\&=\frac{1}{a(\varepsilon)}\left(\mu_1(\varepsilon)\cdot\frac{a(\varepsilon)\xi_1(\varepsilon)+b(\varepsilon)\xi_2(\varepsilon)}{2}+\mu_2(\varepsilon)\cdot\frac{a(\varepsilon)\xi_1(\varepsilon)-b(\varepsilon)\xi_2(\varepsilon)}{2}\right)\\&=\frac{\mu_1(\varepsilon)+\mu_2(\varepsilon)}{2}\cdot\xi_1(\varepsilon)+\nu(\varepsilon)\cdot\frac{\mu_1(\varepsilon)-\mu_2(\varepsilon)}{2}\cdot\xi_2(\varepsilon).\end{aligned}$$

Similarly, we have

$$M(\varepsilon)\xi_2(\varepsilon)=\frac{\mu_1(\varepsilon)-\mu_2(\varepsilon)}{2\nu(\varepsilon)}\cdot\xi_1(\varepsilon)+\frac{\mu_1(\varepsilon)+\mu_2(\varepsilon)}{2}\cdot\xi_2(\varepsilon),\tag{2.9}$$

$$M(\varepsilon)\xi_3(\varepsilon)=\frac{\mu_3(\varepsilon)+\mu_4(\varepsilon)}{2}\cdot\xi_3(\varepsilon)+\bar{\nu}(\varepsilon)\cdot\frac{\mu_3(\varepsilon)-\mu_4(\varepsilon)}{2}\cdot\xi_4(\varepsilon),\tag{2.10}$$

$$M(\varepsilon)\xi_4(\varepsilon)=\frac{\mu_3(\varepsilon)-\mu_4(\varepsilon)}{2\bar{\nu}(\varepsilon)}\cdot\xi_3(\varepsilon)+\frac{\mu_3(\varepsilon)+\mu_4(\varepsilon)}{2}\cdot\xi_4(\varepsilon).\tag{2.11}$$

Since $\langle\xi_j(\varepsilon),\xi_k^*(\varepsilon)\rangle=\delta_{jk}$ by virtue of $W^*(\varepsilon)={}^{\mathrm{t}}W(\varepsilon)^{-1}$, (2.6) follows from (2.8) and (2.9). Moreover, we have

$$M(\varepsilon)W(\varepsilon)=W(\varepsilon)A(\varepsilon),$$

where $A(\varepsilon)$ is a $2n \times 2n$ matrix of the form

$$A(\varepsilon) = \left(\begin{array}{c|c|c} P(\varepsilon) & O & O \\ \hline O & Q(\varepsilon) & O \\ \hline O & O & R(\varepsilon) \end{array}\right),$$

$P(\varepsilon)$ and $Q(\varepsilon)$ are given by

$$P(\varepsilon) = \begin{pmatrix} \dfrac{\mu_1(\varepsilon)+\mu_2(\varepsilon)}{2} & \dfrac{\mu_1(\varepsilon)-\mu_2(\varepsilon)}{2\nu(\varepsilon)} \\ \nu(\varepsilon)\cdot\dfrac{\mu_1(\varepsilon)-\mu_2(\varepsilon)}{2} & \dfrac{\mu_1(\varepsilon)+\mu_2(\varepsilon)}{2} \end{pmatrix},$$

$$Q(\varepsilon) = \begin{pmatrix} \dfrac{\mu_3(\varepsilon)+\mu_4(\varepsilon)}{2} & \dfrac{\mu_3(\varepsilon)-\mu_4(\varepsilon)}{2\bar\nu(\varepsilon)} \\ \bar\nu(\varepsilon)\cdot\dfrac{\mu_3(\varepsilon)-\mu_4(\varepsilon)}{2} & \dfrac{\mu_3(\varepsilon)+\mu_4(\varepsilon)}{2} \end{pmatrix},$$

and $R(\varepsilon)$ is a $(2n-4)\times(2n-4)$ matrix. In addition, it is easy to see that

$$\lim_{\varepsilon\to 0}\frac{\mu_1(\varepsilon)+\mu_2(\varepsilon)}{2} = \lim_{\varepsilon\to 0}\frac{\mu_1(\varepsilon)-\mu_2(\varepsilon)}{2\nu(\varepsilon)} = \rho, \qquad \lim_{\varepsilon\to 0}\nu(\varepsilon)\cdot\frac{\mu_1(\varepsilon)-\mu_2(\varepsilon)}{2} = 0,$$
$$\lim_{\varepsilon\to 0}\frac{\mu_3(\varepsilon)+\mu_4(\varepsilon)}{2} = \lim_{\varepsilon\to 0}\frac{\mu_3(\varepsilon)-\mu_4(\varepsilon)}{2\bar\nu(\varepsilon)} = \rho^{-1}, \qquad \lim_{\varepsilon\to 0}\bar\nu(\varepsilon)\cdot\frac{\mu_3(\varepsilon)-\mu_4(\varepsilon)}{2} = 0$$

because $A(\varepsilon) = W(\varepsilon)^{-1}M(\varepsilon)W(\varepsilon) \to W^{-1}MW = A$ as $\varepsilon \to 0$ by virtue of (2.5).

In a similar manner to the argument in the proof of Lemma 2.1, we have $M(\varepsilon)J^{-1}W^*(\varepsilon) = J^{-1}W^*(\varepsilon)A^*(\varepsilon)$, where

$$A^*(\varepsilon) := {}^{\mathrm t}A(\varepsilon)^{-1} = \left(\begin{array}{c|c|c} {}^{\mathrm t}P(\varepsilon)^{-1} & O & O \\ \hline O & {}^{\mathrm t}Q(\varepsilon)^{-1} & O \\ \hline O & O & {}^{\mathrm t}R(\varepsilon)^{-1} \end{array}\right).$$

Since $\mu_1(\varepsilon)\mu_2(\varepsilon)\mu_3(\varepsilon)\mu_4(\varepsilon) = 1$ by the symplectic property of $M(\varepsilon)$, we have

$$\det P(\varepsilon) = \mu_1(\varepsilon)\mu_2(\varepsilon) = \big(\mu_3(\varepsilon)\mu_4(\varepsilon)\big)^{-1}, \qquad \det Q(\varepsilon) = \mu_3(\varepsilon)\mu_4(\varepsilon) = \big(\mu_1(\varepsilon)\mu_2(\varepsilon)\big)^{-1},$$

so that

$${}^{\mathrm t}P(\varepsilon)^{-1} = \mu_3(\varepsilon)\mu_4(\varepsilon)\begin{pmatrix} \dfrac{\mu_1(\varepsilon)+\mu_2(\varepsilon)}{2} & -\nu(\varepsilon)\cdot\dfrac{\mu_1(\varepsilon)-\mu_2(\varepsilon)}{2} \\ -\dfrac{\mu_1(\varepsilon)-\mu_2(\varepsilon)}{2\nu(\varepsilon)} & \dfrac{\mu_1(\varepsilon)+\mu_2(\varepsilon)}{2} \end{pmatrix},$$

$$
{}^{\mathrm{t}}Q(\varepsilon)^{-1} = \mu_1(\varepsilon)\mu_2(\varepsilon)\begin{pmatrix} \dfrac{\mu_3(\varepsilon)+\mu_4(\varepsilon)}{2} & -\bar{\nu}(\varepsilon)\cdot\dfrac{\mu_3(\varepsilon)-\mu_4(\varepsilon)}{2} \\ -\dfrac{\mu_3(\varepsilon)-\mu_4(\varepsilon)}{2\bar{\nu}(\varepsilon)} & \dfrac{\mu_3(\varepsilon)+\mu_4(\varepsilon)}{2} \end{pmatrix}.
$$

First, we consider the case $\mu_1(\varepsilon)\mu_3(\varepsilon) = \mu_2(\varepsilon)\mu_4(\varepsilon) = 1$ (see Fig. 1(i)). Then we have

$$
{}^{\mathrm{t}}P(\varepsilon)^{-1} = \begin{pmatrix} \dfrac{\mu_3(\varepsilon)+\mu_4(\varepsilon)}{2} & \nu(\varepsilon)\cdot\dfrac{\mu_3(\varepsilon)-\mu_4(\varepsilon)}{2} \\ \dfrac{\mu_3(\varepsilon)-\mu_4(\varepsilon)}{2\nu(\varepsilon)} & \dfrac{\mu_3(\varepsilon)+\mu_4(\varepsilon)}{2} \end{pmatrix},
$$

which yields

$$
\begin{aligned}
M(\varepsilon)J^{-1}\xi_1^*(\varepsilon) &= \frac{\mu_3(\varepsilon)+\mu_4(\varepsilon)}{2}\cdot J^{-1}\xi_1^*(\varepsilon) + \frac{\mu_3(\varepsilon)-\mu_4(\varepsilon)}{2\nu(\varepsilon)}\cdot J^{-1}\xi_2^*(\varepsilon), \\
M(\varepsilon)J^{-1}\xi_2^*(\varepsilon) &= \nu(\varepsilon)\cdot\frac{\mu_3(\varepsilon)-\mu_4(\varepsilon)}{2}\cdot J^{-1}\xi_1^*(\varepsilon) + \frac{\mu_3(\varepsilon)+\mu_4(\varepsilon)}{2}\cdot J^{-1}\xi_2^*(\varepsilon).
\end{aligned}
$$

Comparing these equalities with (2.10) and (2.11), we obtain $J^{-1}\xi_1^*(\varepsilon) = -\xi_4(\varepsilon)$ and $J^{-1}\xi_2^*(\varepsilon) = \xi_3(\varepsilon)$ provided $\bar{\nu}(\varepsilon) = -\nu(\varepsilon)$, i.e., $\sigma(\varepsilon) = \pm i$. In other words, if and only if $\sigma(\varepsilon) = \pm i$, we can obtain $J^{-1}\xi_1^*(\varepsilon) = -\xi_4(\varepsilon)$ and $J^{-1}\xi_2^*(\varepsilon) = \xi_3(\varepsilon)$ which must be true because of Lemma 2.1 and (2.5). Similarly, we have $J^{-1}\xi_3^*(\varepsilon) = -\xi_2(\varepsilon)$ and $J^{-1}\xi_4^*(\varepsilon) = \xi_1(\varepsilon)$ provided $\sigma(\varepsilon) = \pm i$. Since $\sigma(\varepsilon)$ is continuous in ε, we see that (2.7) holds when $\sigma(\varepsilon) \equiv i$ or $\sigma(\varepsilon) \equiv -i$ in the case of $\mu_1(\varepsilon)\mu_3(\varepsilon) = \mu_2(\varepsilon)\mu_4(\varepsilon) = 1$.

In a similar manner to the above case, we can verify that (2.7) holds when $\sigma(\varepsilon) \equiv 1$ or $\sigma(\varepsilon) \equiv -1$ in the case of $\mu_1(\varepsilon)\mu_4(\varepsilon) = \mu_2(\varepsilon)\mu_3(\varepsilon) = 1$ (see Fig. 1(ii)).

Thus, if we define $\{\xi_j(\varepsilon)\}$ by

$$
\begin{aligned}
&\xi_1(\varepsilon) = \tilde{\xi}_1(\varepsilon), \qquad \xi_3(\varepsilon) = \tilde{\xi}_3(\varepsilon), \\
&\xi_2(\varepsilon) = \begin{cases} \|\eta_2\|\dfrac{\hat{\xi}_1(\varepsilon)-\hat{\xi}_2(\varepsilon)}{\|\hat{\xi}_1(\varepsilon)-\hat{\xi}_2(\varepsilon)\|} & \text{if } \mu_1(\varepsilon)\mu_4(\varepsilon) = \mu_2(\varepsilon)\mu_3(\varepsilon) = 1, \\ -i\|\eta_2\|\dfrac{\hat{\xi}_1(\varepsilon)-\hat{\xi}_2(\varepsilon)}{\|\hat{\xi}_1(\varepsilon)-\hat{\xi}_2(\varepsilon)\|} & \text{if } \mu_1(\varepsilon)\mu_3(\varepsilon) = \mu_2(\varepsilon)\mu_4(\varepsilon) = 1, \end{cases}
\end{aligned}
$$

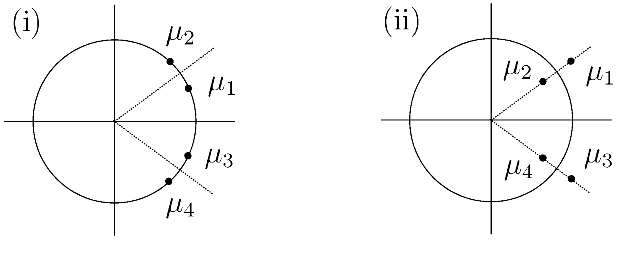

Fig. 1. The position of multipliers.

$$\xi_4(\varepsilon)=\begin{cases}\|\eta_4\|\dfrac{\hat{\xi}_3(\varepsilon)-\hat{\xi}_4(\varepsilon)}{\|\hat{\xi}_3(\varepsilon)-\hat{\xi}_4(\varepsilon)\|} & \text{if } \mu_1(\varepsilon)\mu_4(\varepsilon)=\mu_2(\varepsilon)\mu_3(\varepsilon)=1,\\ i\|\eta_4\|\dfrac{\hat{\xi}_3(\varepsilon)-\hat{\xi}_4(\varepsilon)}{\|\hat{\xi}_3(\varepsilon)-\hat{\xi}_4(\varepsilon)\|} & \text{if } \mu_1(\varepsilon)\mu_3(\varepsilon)=\mu_2(\varepsilon)\mu_4(\varepsilon)=1,\end{cases}$$

$\xi_1(\varepsilon)$, $\xi_2(\varepsilon)$, $\xi_3(\varepsilon)$ and $\xi_4(\varepsilon)$ are differentiable in ε on $(-\delta,\delta)$ for some $\delta>0$, and (2.4)–(2.7) hold. □

Here, let us define a basis of solutions of $Jx_t=H(t,0)x$ by

$$\eta_j(t):=M(t,0)\eta_j,\quad j=1,2,\dots,2n,$$

where $\{\eta_j\}$ are as in Lemma 2.1. Notice that $\eta_3(t)=\overline{\eta_1(t)}$ and $\eta_4(t)=\overline{\eta_2(t)}$ by virtue of $\eta_3=\bar{\eta}_1$ and $\eta_4=\bar{\eta}_2$. Furthermore, we define a basis of $Jx_t=H(t,\varepsilon)x$ by

$$\xi_j(t,\varepsilon):=M(t,\varepsilon)\xi_j(\varepsilon),\quad j=1,2,\dots,2n,$$

where $\{\xi_j(\varepsilon)\}$ are as in Lemma 2.2. Then it follows from (2.6) that

$$\mu_1(\varepsilon)+\mu_2(\varepsilon)=\langle\xi_1(T,\varepsilon),\xi_1^*(\varepsilon)\rangle+\langle\xi_2(T,\varepsilon),\xi_2^*(\varepsilon)\rangle. \tag{2.12}$$

Since $\xi_j(t,\varepsilon)$ is a solution of (1.1), differentiating $J\partial_t\xi_j(t,\varepsilon)=H(t,\varepsilon)\xi_j(t,\varepsilon)$ with respect to ε and setting $\varepsilon=0$, we obtain

$$J\frac{d}{dt}\partial_\varepsilon\xi_j(t,0)=H(t,0)\partial_\varepsilon\xi_j(t,0)+\frac{\partial H}{\partial\varepsilon}(t,0)\xi_j(t,0).$$

Hence $\partial_\varepsilon\xi_j(t,0)$ is written as

$$\partial_\varepsilon\xi_j(t,0)=z_j(t)+\sum_{k=1}^{2n}p_{jk}\eta_k(t), \tag{2.13}$$

where $z_j(t)$ is a unique solution of

$$J\frac{d}{dt}z_j=H(t,0)z_j+\frac{\partial H}{\partial\varepsilon}(t,0)\xi_j(t,0),\quad z_j(0)=0, \tag{2.14}$$

and p_{jk} $(j,k=1,2,\dots,2n)$ are certain constants.

Lemma 2.3.

$$\frac{d}{d\varepsilon}\big(\mu_1(\varepsilon)+\mu_2(\varepsilon)\big)\Big|_{\varepsilon=0}=\langle z_1(T),\eta_1^*\rangle+\langle z_2(T),\eta_2^*\rangle.$$

Proof. Differentiating (2.12) with respect to ε, we obtain

$$\frac{d}{d\varepsilon}\big(\mu_1(\varepsilon)+\mu_2(\varepsilon)\big) = \big\langle \partial_\varepsilon \xi_1(T,\varepsilon),\ \xi_1^*(\varepsilon)\big\rangle + \big\langle \xi_1(T,\varepsilon),\ \partial_\varepsilon \xi_1^*(\varepsilon)\big\rangle + \big\langle \partial_\varepsilon \xi_2(T,\varepsilon),\ \xi_2^*(\varepsilon)\big\rangle + \big\langle \xi_2(T,\varepsilon),\ \partial_\varepsilon \xi_2^*(\varepsilon)\big\rangle.$$

We will compute the right-hand side termwise at $\varepsilon = 0$.

First, it follows from (2.5) and (2.13) that

$$\begin{aligned}\big\langle \partial_\varepsilon \xi_1(T,\varepsilon), \xi_1^*(\varepsilon)\big\rangle\big|_{\varepsilon=0} &= \big\langle z_1(T),\ \xi_1^*(0)\big\rangle + \sum_{k=1}^{2n} p_{1k}\big\langle \eta_k(T), \xi_1^*(0)\big\rangle \\ &= \big\langle z_1(T), \eta_1^*\big\rangle + \sum_{k=1}^{2n} p_{1k}\big\langle \eta_k(T), \eta_1^*\big\rangle.\end{aligned}$$

Then, by $\eta_1(T) = M(T,0)\eta_1 = \rho\eta_1$ and $\eta_2(T) = M(T,0)\eta_2 = \rho\eta_1 + \rho\eta_2$, we obtain

$$\big\langle \partial_\varepsilon \xi_1(T,\varepsilon),\ \xi_1^*(\varepsilon)\big\rangle\big|_{\varepsilon=0} = \big\langle z_1(T), \eta_1^*\big\rangle + \rho(p_{11}+p_{12}). \tag{2.15}$$

Similarly, we have

$$\big\langle \partial_\varepsilon \xi_2(T,\varepsilon),\ \xi_2^*(\varepsilon)\big\rangle\big|_{\varepsilon=0} = \big\langle z_2(T), \eta_2^*\big\rangle + \rho p_{22}. \tag{2.16}$$

Next, it follows from (2.4) and (2.7) that

$$\big\langle \xi_1(T,\varepsilon), \partial_\varepsilon \xi_1^*(\varepsilon)\big\rangle\big|_{\varepsilon=0} = \big\langle \xi_1(T,\varepsilon), \partial_\varepsilon\big(-J\xi_4(\varepsilon)\big)\big\rangle\big|_{\varepsilon=0} = \big\langle J\xi_1(T,\varepsilon), \overline{\partial_\varepsilon \xi_2(\varepsilon)}\,\big\rangle\big|_{\varepsilon=0}.$$

Since

$$\begin{gathered}\xi_1(T,\varepsilon)|_{\varepsilon=0} = M(T,\varepsilon)\xi_1(0,\varepsilon)|_{\varepsilon=0} = M(T,0)\eta_1 = \rho\eta_1,\\ \partial_\varepsilon \xi_2(\varepsilon)|_{\varepsilon=0} = \partial_\varepsilon \xi_2(0,0) = \sum_{k=1}^{2n} p_{2k}\eta_k,\end{gathered}$$

$\bar\eta_4^* = \eta_2^*$ and $\eta_4^* = J\eta_1$ by Lemma 2.1, we obtain

$$\begin{aligned}\big\langle \xi_1(T,\varepsilon),\ \partial_\varepsilon \xi_1^*(\varepsilon)\big\rangle\big|_{\varepsilon=0} &= \rho\sum_{k=1}^{2n}\langle J\eta_1, \overline{p_{2k}\eta_k}\,\rangle = \rho\sum_{k=1}^{2n}\langle \eta_4^*, \overline{p_{2k}\eta_k}\,\rangle \\ &= \rho\sum_{k=1}^{2n}\langle p_{2k}\eta_k, \bar\eta_4^*\rangle = \rho\sum_{k=1}^{2n} p_{2k}\langle \eta_k, \eta_2^*\rangle = \rho p_{22}.\end{aligned} \tag{2.17}$$

Similarly, we have

$$\big\langle \xi_2(T,\varepsilon),\ \partial_\varepsilon \xi_2^*(\varepsilon)\big\rangle\big|_{\varepsilon=0} = \rho p_{11} - \rho p_{12}. \tag{2.18}$$

Thus, by (2.15)–(2.18), we obtain

$$\begin{aligned}
&\frac{d}{d\varepsilon}\big(\mu_1(\varepsilon)+\mu_2(\varepsilon)\big)\big|_{\varepsilon=0}\\
&\quad=\big\langle z_1(T),\eta_1^*\big\rangle+\rho(p_{11}+p_{12})+\rho p_{22}+\big\langle z_2(T),\eta_2^*\big\rangle+\rho p_{22}+\rho p_{11}-\rho p_{12}\\
&\quad=\big\langle z_1(T),\eta_1^*\big\rangle+\big\langle z_2(T),\eta_2^*\big\rangle+2\rho(p_{11}+p_{22}).
\end{aligned}$$

Now, it suffices to show $p_{11}+p_{22}=0$. Differentiating $\langle \xi_1(\varepsilon),\xi_1^*(\varepsilon)\rangle=1$ in ε, we have

$$\big\langle \partial_\varepsilon\xi_1(\varepsilon),\xi_1^*(\varepsilon)\big\rangle+\big\langle \xi_1(\varepsilon),\partial_\varepsilon\xi_1^*(\varepsilon)\big\rangle=0.$$

Here, since

$$\partial_\varepsilon\xi_1(\varepsilon)|_{\varepsilon=0}=\partial_\varepsilon\xi_1(0,0)=\sum_{k=1}^{2n}p_{1k}\eta_k,$$

we have

$$\big\langle \partial_\varepsilon\xi_1(\varepsilon),\ \xi_1^*(\varepsilon)\big\rangle\big|_{\varepsilon=0}=\sum_{k=1}^{2n}p_{1k}\langle\eta_k,\eta_1^*\rangle=p_{11}.$$

Moreover, in a similar manner to the argument as applied to (2.17), we have

$$\big\langle \xi_1(\varepsilon),\ \partial_\varepsilon\xi_1^*(\varepsilon)\big\rangle=\sum_{k=1}^{2n}\langle J\eta_1,\overline{p_{2k}\eta_k}\rangle=p_{22}.$$

Thus we obtain $p_{11}+p_{22}=0$. This completes the proof. □

Proposition 2.4. *Assume that $\rho\neq\pm1$ is an indefinite multiplier and that $\{\eta_j\}$ are as in Lemma* 2.1. *Then*

$$\frac{d}{d\varepsilon}\big(\mu_1(\varepsilon)+\mu_2(\varepsilon)\big)\Big|_{\varepsilon=0}=-\bar\rho\int_0^T\Big\langle\frac{\partial H}{\partial\varepsilon}(t,0)\eta_1(t),\overline{\eta_1(t)}\Big\rangle dt.$$

Proof. It follows from (2.14) that

$$\begin{aligned}
&\int_0^T\Big\langle\frac{\partial H}{\partial\varepsilon}(t,0)\eta_1(t),\bar\eta_2(t)\Big\rangle dt\\
&\quad=\int_0^T\Big\langle\frac{\partial H}{\partial\varepsilon}(t,0)\xi_1(t,0),\eta_4(t)\Big\rangle dt=\int_0^T\Big\langle J\frac{d}{dt}z_1-H(t,0)z_1,\eta_4(t)\Big\rangle dt
\end{aligned}$$

$$= \big[\langle Jz_1(t), \eta_4(t)\rangle\big]_0^T - \int_0^T \left\langle z_1(t), {}^{\mathrm{t}}J\frac{d}{dt}\eta_4(t)\right\rangle dt - \int_0^T \langle z_1(t), {}^{\mathrm{t}}H(t,0)\eta_4(t)\rangle\, dt$$

$$= \langle Jz_1(T), \eta_4(T)\rangle + \int_0^T \left\langle z_1(t), J\frac{d}{dt}\eta_4(t) - H(t,0)\eta_4(t)\right\rangle dt$$

$$= \langle Jz_1(T), \eta_4(T)\rangle.$$

Here, since $\eta_4(T) = M(T,0)\eta_4 = \rho^{-1}\eta_3 + \rho^{-1}\eta_4 = \bar{\rho}\eta_3 + \bar{\rho}\eta_4$, and $\eta_2^* = J\eta_3$ and $\eta_1^* = -J\eta_4$ by Lemma 2.1, we have

$$\langle Jz_1(T), \eta_4(T)\rangle = -\rho\langle z_1(T), J\eta_3\rangle - \rho\langle z_1(T), J\eta_4\rangle = -\rho\langle z_1(T), \eta_2^*\rangle + \rho\langle z_1(T), \eta_1^*\rangle.$$

Hence we obtain

$$\int_0^T \left\langle \frac{\partial H}{\partial \varepsilon}(t,0)\eta_1(t), \bar{\eta}_2(t)\right\rangle dt = -\rho\langle z_1(T), \eta_2^*\rangle + \rho\langle z_1(T), \eta_1^*\rangle.$$

Similarly, we have

$$\int_0^T \left\langle \frac{\partial H}{\partial \varepsilon}(t,0)\eta_2(t), \bar{\eta}_1(t)\right\rangle dt = -\rho\langle z_2(T), \eta_2^*\rangle \quad \text{and}$$

$$\int_0^T \left\langle \frac{\partial H}{\partial \varepsilon}(t,0)\eta_1(t), \bar{\eta}_1(t)\right\rangle dt = -\rho\langle z_1(T), \eta_2^*\rangle.$$

Using these equalities and noting that $\partial_\varepsilon H(t,0)$ is a real symmetric matrix, we obtain

$$\begin{aligned}
\langle z_1(T), \eta_1^*\rangle + \langle z_2(T), \eta_2^*\rangle &= \rho^{-1}\left\{\int_0^T \left\langle \frac{\partial H}{\partial \varepsilon}(t,0)\eta_1(t), \bar{\eta}_2(t)\right\rangle dt - \int_0^T \left\langle \frac{\partial H}{\partial \varepsilon}(t,0)\eta_2(t), \bar{\eta}_1(t)\right\rangle dt \right.\\
&\qquad \left. - \int_0^T \left\langle \frac{\partial H}{\partial \varepsilon}(t,0)\eta_1(t), \bar{\eta}_1(t)\right\rangle dt\right\}\\
&= -\rho^{-1}\int_0^T \left\langle \frac{\partial H}{\partial \varepsilon}(t,0)\eta_1(t), \overline{\eta_1(t)}\right\rangle dt.
\end{aligned}$$

By Lemma 2.3 and $\rho^{-1} = \bar{\rho}$, the proof is complete. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. First we consider the case where $\rho \neq \pm 1$. If we take $\{\eta_j\}$ as in Lemma 2.1, we see from Proposition 2.4 and $\langle J\eta_1, \bar{\eta}_2\rangle = \langle J\eta_1, \eta_4\rangle = 1$ that (1.7) is true. Here we consider arbitrariness of generalized eigenvectors η_1 and η_2. Replacing η_1 by $c_1\eta_1$, η_2 must be replaced by $c_1\eta_2$ because $M(T,0)\eta_2 = \rho\eta_2 + \rho\eta_1$, and $\eta_1(t), \bar{\eta}_1(t)$ are replaced by $c_1\eta_1(t), c_1\bar{\eta}_1(t)$, respectively. Hence we may use $\langle J\eta_1, \bar{\eta}_2\rangle$ as a normalizing constant. On the other hand, by replacing η_2 by $\eta_2 + c_2\eta_1$ as (2.2), the constant c_2 does not affect the value of $\langle J\eta_1, \bar{\eta}_2\rangle$ because $\langle J\eta_1, \bar{\eta}_1\rangle = \langle J\eta_1, \eta_3\rangle = 0$ by Lemma 2.1. Thus we see that (1.7) holds for any choice of η_1 and η_2.

Next, let us consider the case where $\rho = \pm 1$. In this case, we have $MW = WA$ and $M^*W^* = W^*A^*$, where

$$A = \left(\begin{array}{cc|c} \rho & \rho & \\ 0 & \rho & O \\ \hline \multicolumn{2}{c|}{O} & B \end{array}\right) \quad \text{and} \quad A^* = {}^{\mathrm{t}}A^{-1} = \left(\begin{array}{cc|c} \rho^{-1} & 0 & \\ -\rho^{-1} & \rho^{-1} & O \\ \hline \multicolumn{2}{c|}{O} & {}^{\mathrm{t}}B^{-1} \end{array}\right),$$

so that $\eta_1^* = -J\eta_2$ and $\eta_2^* = J\eta_1$ under the assumption of $\langle J\eta_1, \eta_2\rangle = 1$. It should be noted that η_1 and η_2 are real. In this case, we can prove an analogous result as Lemma 2.2; (2.5), (2.6), and $\xi_1^*(\varepsilon) = -J\xi_2(\varepsilon)$ and $\xi_2^*(\varepsilon) = J\xi_1(\varepsilon)$ (cf. (2.7)). Using these results, we can derive (1.7) for $\rho = \pm 1$ in a similar manner to the above argument. Thus the proof of Theorem 1.2 is complete. □

3. Applications

In this section, we give several examples that demonstrate usefulness of the formulae (1.7) and (1.8). Although it is expected they are basic tools for the study of various problems of Hamiltonian systems, we reconsider well known and fundamental results from a viewpoint of direct applications of the formulae (1.7) and (1.8).

3.1. An extension of the Krein–Lyubarskii theorem

System (1.1) is said to be strongly stable if all the solutions of (1.1) are bounded on $(-\infty, +\infty)$, and this property is preserved by any small perturbation. As for strong stability of (1.1), Krein, Gelfand and Lidsky showed the following fundamental result.

Theorem 3.1 *(Krein–Gelfand–Lidsky). System* (1.1) *is strongly stable if and only if all the multipliers of* (1.1) *lie on the unit circle and are definite.*

This theorem motivates us to study the case where all the multipliers are not definite. Krein and Lyubarskii discussed the behavior of multipliers for (1.1) with $H(t,\varepsilon) = H(t) + \varepsilon Q(t)$, where $Q(t)$ is positive definite. According to [16, Theorem I, pp. 208], under an assumption as in Theorem 1.2, they showed that in one of the intervals $-\varepsilon_0 < \varepsilon < 0$ or $0 < \varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$, (1.1) has multipliers in a neighborhood of ρ which are not on the unit circle. However, their result does not tell us about in which interval the multipliers are not on the unit circle. The following theorem gives an answer to this problem for general Hamiltonian systems defined by (1.1) under the assumption of Theorem 1.2, and the proof is quite simple.

Theorem 3.2. *Suppose that the assumption of Theorem* 1.2 *is satisfied. If*

$$\mathrm{Re}\left\{\frac{\bar\rho}{\langle J\eta_1,\bar\eta_2\rangle}\int_0^T\left\langle\frac{\partial H}{\partial\varepsilon}(t,0)\eta_1(t),\overline{\eta_1(t)}\right\rangle dt\right\}>0\qquad(<0)\qquad \textit{for } \rho\neq\pm i,$$
$$\mathrm{Im}\left\{\frac{\bar\rho}{\langle J\eta_1,\bar\eta_2\rangle}\int_0^T\left\langle\frac{\partial H}{\partial\varepsilon}(t,0)\eta_1(t),\overline{\eta_1(t)}\right\rangle dt\right\}>0\qquad(<0)\qquad \textit{for } \rho=\pm i,$$

then the following holds:

(1) $\mu_1(\varepsilon)$ *and* $\mu_2(\varepsilon)$ *are on the unit circle if* $\varepsilon>0$ (<0).
(2) $\mu_1(\varepsilon)$ *and* $\mu_2(\varepsilon)$ *are off the unit circle if* $\varepsilon<0$ (>0).

Consequently, (1.1) *is strongly stable if* $\varepsilon>0$ (<0) *when all other multipliers except* $\mu_3(\varepsilon)=\bar\mu_1(\varepsilon)$ *and* $\mu_4(\varepsilon)=\bar\mu_2(\varepsilon)$ *lie on the unit circle and are definite.*

Proof. We consider the case $\rho\neq\pm1$ and $\rho\neq\pm i$ because the case $\rho=\pm1$ or $\rho=\pm i$ can be easily and similarly treated. Since ρ $(=\mu_1(0)=\mu_2(0))$ is a degenerate eigenvalue whose algebraic multiplicity is equal to two, by noting the property that μ^{-1} and $\bar\mu$ are eigenvalues whenever μ is, the following must hold when $\varepsilon\neq0$:

(i) $\mu_1(\varepsilon)$ and $\mu_2(\varepsilon)$ are on the unit circle, and $\mu_1(\varepsilon)\approx\rho e^{\pm ic\sqrt{|\varepsilon|}}$ and $\mu_2(\varepsilon)\approx\rho e^{\mp ic\sqrt{|\varepsilon|}}$ for sufficiently small ε with some $c>0$ (see Fig. 1(i)).
(ii) $\mu_1(\varepsilon)$ and $\mu_2(\varepsilon)$ are off the unit circle, and on the half-line $\ell=\{z\in\mathbf{C}\mid\arg z=\arg\rho\}$ with $|\mu_1(\varepsilon)\mu_2(\varepsilon)|=1$ (see Fig. 1(ii)).

Let us define

$$\theta(\varepsilon):=\mu_1(\varepsilon)+\mu_2(\varepsilon)+\mu_3(\varepsilon)+\mu_4(\varepsilon)=\mu_1(\varepsilon)+\mu_2(\varepsilon)+\overline{\mu_1(\varepsilon)+\mu_2(\varepsilon)}\in\mathbf{R}.$$

Then it is easy to see that $\theta(0)=4\,\mathrm{Re}\,\rho$, $\theta(\varepsilon)<4\,\mathrm{Re}\,\rho$ in case (i), and $\theta(\varepsilon)>4\,\mathrm{Re}\,\rho$ in case (ii). On the other hand, it follows from Theorem 1.2 that

$$\frac{d}{d\varepsilon}\theta(\varepsilon)\Big|_{\varepsilon=0}=-2\,\mathrm{Re}\left\{\frac{\bar\rho}{\langle J\eta_1,\bar\eta_2\rangle}\int_0^T\left\langle\frac{\partial H}{\partial\varepsilon}(t,0)\eta_1(t),\overline{\eta_1(t)}\right\rangle dt\right\}.$$

Thus the assertion holds. □

3.2. The Eckhaus instability in dissipative systems

Spatially periodic patterns are observed in various natural phenomena such as thermal convection, biological morphology, crystal growth and so on. Here we consider a family of spatially periodic patterns in dissipative systems with gradient/skew-gradient structure introduced in [10,17], which covers the Swift–Hohenberg equation and some reaction–diffusion systems of activator–inhibitor type.

Let us consider an n-component system on $\mathbf{R}$

$$Tu_t = Du_{xx} + Q\nabla_u F(u), \quad u(x,t) = {}^{\mathrm{t}}(u_1, u_2, \ldots, u_n) \in \mathbf{R}^n, \tag{3.1}$$

where T is a nonnegative diagonal matrix, Q is a symmetric matrix with $Q^2 = I_n$, D is a non-singular matrix, and $F = F(u) : \mathbf{R}^n \to \mathbf{R}$ is a smooth function. In addition, we assume that ${}^{\mathrm{t}}(QT) = QT$ and ${}^{\mathrm{t}}(QD) = QD$. In this case, (3.1) is called a gradient/skew-gradient dissipative system. For example, reaction–diffusion system of activator–inhibitor type

$$\tau_1 u_t = d_1 u_{xx} + \alpha u - u^3 - v, \qquad \tau_2 v_t = d_2 v_{xx} + u - \gamma v \tag{3.2}$$

is a skew-gradient system because it is rewritten as (3.1) by setting $(u_1, u_2) = (u, v)$ and

$$T = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \qquad D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$F = F(u,v) = \frac{1}{2}\alpha u^2 - \frac{1}{4}u^4 - uv + \frac{1}{2}\gamma v^2.$$

Moreover, the Swift–Hohenberg equation

$$u_t = \mu u - (1 + \partial_{xx})^2 u - u^3 \tag{3.3}$$

is a gradient system. In fact, noting $v = u + u_{xx}$, we find that (3.3) is rewritten as (3.1) by setting $(u_1, u_2) = (u, v)$ and

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$F = F(u,v) = \frac{\mu}{2}u^2 - \frac{1}{4}u^4 - uv + \frac{1}{2}v^2.$$

It is well known that (3.2) and (3.3) exhibit stationary patterns with spatially periodic structure [9,10]. Let $u = \varphi(x;\ell)$ be a family of spatially periodic stationary solutions of (3.1) parametrized by its wavelength ℓ, that is, $\varphi(x,\ell)$ satisfies

$$\begin{cases} D\varphi_{xx}(x;\ell) + Q\nabla F\big(\varphi(x;\ell)\big) = 0, \\ \varphi(x;\ell) \equiv \varphi(x+\ell;\ell). \end{cases} \tag{3.4}$$

In order to study linear stability of stationary solutions in (3.1), we consider the linearized eigenvalue problem

$$\lambda T W = DW_{xx} + Q\nabla^2 F\big(\varphi(x;\ell)\big)W. \tag{3.5}$$

Setting $W_x = Z$, (3.5) can be rewritten in the form of (1.1) as

$$JV_x = H(x;\ell)V + \lambda NV, \tag{3.6}$$

where

$$V = {}^{\mathrm{t}}(W, Z), \qquad J = \begin{pmatrix} 0 & QD \\ -QD & 0 \end{pmatrix},$$

$$H(x;\ell) = \begin{pmatrix} -\nabla^2 F(\varphi(x;\ell)) & 0 \\ 0 & -QD \end{pmatrix}, \qquad N = \begin{pmatrix} QT & 0 \\ 0 & 0 \end{pmatrix}.$$

The stationary solution $u = \varphi(x;\ell)$ is unstable if (3.5) has a bounded solution for some $\operatorname{Re}\lambda > 0$, that is, (3.6) has multipliers on the unit circle.

It is easy to see that (3.6) has an indefinite multiplier 1 for $\lambda = 0$. Indeed, differentiating (3.4) in x, we see that $\varphi_x(x;\ell)$ is a bounded solution of (3.5) for $\lambda = 0$, so that (3.6) has a multiplier 1 for $\lambda = 0$. Similarly, differentiating (3.4) in ℓ, and by using the periodicity of $\varphi(x;\ell)$, we see that 1 is an indefinite multiplier. More precisely, for $\eta_1 = {}^{\mathrm{t}}(\varphi_x(0;\ell), \varphi_{xx}(0,\ell))$ and $\eta_2 = {}^{\mathrm{t}}(-\varphi_\ell(0;\ell), -\varphi_{x\ell}(0,\ell))$, we have

$$M(\ell)(\eta_1\ \eta_2) = (\eta_1\ \eta_2)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

where $M(\ell)$ is the monodromy matrix of (3.6) for $\lambda = 0$. Let $\mu_1(\lambda)$ and $\mu_2(\lambda)$ be multipliers with $\mu_1(0) = \mu_2(0) = 1$. We regard λ as a real perturbation parameter, and apply Theorem 1.2 expressed in the form of (1.8). Noting $\eta_1(x) = {}^{\mathrm{t}}(\varphi_x(x;\ell), \varphi_{xx}(x,\ell))$ and $\eta_2(x) = {}^{\mathrm{t}}(-\varphi_\ell(x;\ell), -\varphi_{x\ell}(x,\ell))$, we have

$$\frac{d}{d\lambda}\theta(\lambda)\Big|_{\lambda=0} = -\ell\frac{\int_0^\ell \langle N\eta_1(x), \eta_1(x)\rangle\,dx}{\int_0^\ell \langle J\eta_1(x), \eta_2(x)\rangle\,dx},$$

where $\theta(\lambda) = \mu_1(\lambda) + \mu_2(\lambda)$. Direct calculation yields that

$$\int_0^\ell \big\langle N\eta_1(x), \eta_1(x)\big\rangle\,dx = \int_0^\ell \big\langle QT\varphi_x(x;\ell),\ \varphi_x(x;\ell)\big\rangle\,dx := I(\ell),$$

$$\begin{aligned}\int_0^\ell \big\langle J\eta_1(x), \eta_2(x)\big\rangle\,dx &= \int_0^\ell \Big(\big\langle QD\varphi_x(x;\ell), \varphi_{x\ell}(x;\ell)\big\rangle - \big\langle QD\varphi_{xx}(x;\ell), \varphi_\ell(x;\ell)\big\rangle\Big)\,dx \\ &= -\big[\big\langle QD\varphi_x(x;\ell), \varphi_\ell(x;\ell)\big\rangle\big]_0^\ell \\ &\quad + \int_0^\ell \Big(\big\langle QD\varphi_x(x;\ell), \varphi_{x\ell}(x;\ell)\big\rangle + \big\langle QD\varphi_x(x;\ell), \varphi_{x\ell}(x;\ell)\big\rangle\Big)\,dx\end{aligned}$$

$$= \big\langle QD\varphi_x(\ell;\ell), \varphi_\ell(\ell;\ell)\big\rangle + 2\int_0^\ell \big\langle QD\varphi_x(x;\ell), \varphi_{x\ell}(x;\ell)\big\rangle\,dx$$

$$= \frac{d}{d\ell}\int_0^\ell \big\langle QD\varphi_x(x;\ell), \varphi_x(x;\ell)\big\rangle\,dx := \frac{d}{d\ell}K(\ell)$$

by using $\varphi_x(\ell;\ell) = \varphi_x(0;\ell)$, $\varphi_\ell(\ell;\ell) = \varphi_\ell(0;\ell) - \varphi_x(0;\ell)$ and ${}^{\mathrm{t}}(QD) = QD$.

Since $\theta(\lambda)$ is real, and $\theta(\lambda) < \theta(0) = 2$ when $\mu_1(\lambda)$ and $\mu_2(\lambda)$ are on the unit circle, (3.6) has multipliers on the unit circle for sufficiently small $\lambda > 0$ if $I(\ell)$ and $dK(\ell)/d\ell$ have the same sign. In other words, when $I(\ell)$ and $dK(\ell)/d\ell$ have the same sign, (3.5) has a bounded solution for some $\operatorname{Re}\lambda > 0$, which implies that $\varphi(x;\ell)$ is unstable. This result is known as the Eckhaus instability criterion [5], that is also obtained in [10].

As seen in the above, it is well recognized that indefinite multipliers play a crucial role in the stability analysis of nonlinear phenomena [3,10,11,18]. For example, [3] discussed stability of periodic solutions due to the Hamiltonian–Hopf bifurcation [12,13,15]. The above result has the same spirit in [3], however, our argument is totally different from [3], in which the analysis of Floquet exponent corresponding to an indefinite multiplier 1 of monodromy matrix of Hamiltonian systems was carried out in the framework of bifurcation theory.

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